Math 2050, quick note of Week 3

1. Sequence and the convergence

We want to study the behaviour of sequence of real numbers, $\{a_n\}_{n=1}^{\infty}$. We want to study the concept of "limit" when $n \to +\infty$.

Definition 1.1. *Given a sequence of real number $\{a_n\}_{n=1}^{\infty}$.*

(i) *$\{a_n\}_{n=1}^{\infty}$ is said to be convergent to $a \in \mathbb{R}$ if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that for all $n > N$, $|a - a_n| < \varepsilon$. In this case, we will write $\lim_{n\to+\infty} a_n = a$ or "$a_n \to a$ as $n \to +\infty$".*

(ii) *We say that $\{a_n\}_{n=1}^{\infty}$ is convergent if there is $a \in \mathbb{R}$ such that $\lim_{n\to+\infty} a_n = a$.*

(iii) *e say that $\{a_n\}_{n=1}^{\infty}$ is divergent if it is not convergent.*

Remark 1.1. In word (roughly speaking), the definition of convergent means that we can control the error ε as much as we wish as long as we consider sufficiently "late" element.

It is sometimes geometrically convenient to use

$$a_n \in V_\varepsilon(a) = \{x : |x - a| < \varepsilon\}$$

to emphasis that a_n is close to a with error at most ε.

To determine the convergence, it is only important to consider large index n. The following Sandwich Theorem illustrate this fact.

Theorem 1.1. *Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $\lim_{n\to+\infty} a_n = 0$, $x, C \in \mathbb{R}, m \in \mathbb{N}$ and $\{x_n\}_{n=1}^{\infty}$ is a sequence such that for all $n > m$, we have*

$$|x - x_n| \le C a_n,$$

then we have $\lim_{n\to+\infty} x_n = x$.

We here give an example which used some common trick in analysis. (see more from the textbook)

Question 1.1. *Show that*

$$\lim_{n\to+\infty} \frac{n^2}{3^n} = 0.$$

Answer. Before we fix ε, let us do some estimate to simplify the question. For $n > 5$, we have

$$3^n = (1+2)^n \ge C_3^n 2^3 = n(n-1)(n-2) \cdot \frac{4}{3}. \tag{1.1}$$

Now, we used the fact that $n > 5$ to show that

$$\frac{n^2}{3^n} < \frac{n^2}{n(n-1)(n-2)} \leq \frac{n^2}{n(n-\frac{n}{2})^2} = \frac{4}{n}.$$

Since $\lim_{n\to+\infty} 1/n = 0$ by Archimedean properties: For all $\varepsilon > 0$, there is N such that

$$\frac{1}{N} < \varepsilon.$$

And hence for all $n > N$, $n^{-1} < \varepsilon$. Now, we may apply Sandwich Theorem with $m = 5$, $C = 4$ and $a_n = 1/n$ to deduce the answer. □

Using the above method, one can actually prove the following:

$$\lim_{n\to+\infty} \frac{P(n)}{(1+a)^n} = 0$$

for any polynomial $P(x)$ and $a > 0$ (Try it!).

We have some simply criterion for convergence.

Theorem 1.2. *Suppose* $\{x_n\}_{n=1}^{\infty}$ *is a convergent sequence, then* $\{x_n\}_{n=1}^{\infty}$ *is bounded.*

Important consequence: (!!!) Equivalently, if a sequence is unbounded, then the sequence is divergent! We will go back to this later.

The algebra operation is preserved under limiting process.

Theorem 1.3. *Suppose* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are two sequence of real number which* $\lim_{n\to+\infty} x_n = x$ *and* $\lim_{n\to+\infty} y_n = y$*. Then we have*

(1) $\lim_{n\to+\infty} x_n + y_n = x + y$;
(2) $\lim_{n\to+\infty} x_n - y_n = x - y$;
(3) $\lim_{n\to+\infty} x_n \cdot y_n = xy$;
(4) $\lim_{n\to+\infty} \frac{x_n}{y_n} = xy^{-1}$ *if* $y \neq 0$.